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Inversive Planes Satisfying the Bundle Theorem

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It is proved that all infinite inversive planes which satisfy the bundle theorem are egglike (ovoidal). For finite inversive planes this was previously proved by the author.

1. INTRODUCTION

An *inversive plane* is an incidence structure (see Section 2) $\mathscr{I} = (\mathscr{O}, \mathscr{C})$ (elements of $\mathscr{C}$ are called circles) which satisfies:

(0) Circles are nonempty.

(1) For each $P \in \mathscr{O}$, $\mathscr{I}_p$ is an affine plane (where $\mathscr{I}_P$ is the "internal" structure whose points are the points of $\mathscr{O}$ other than P, whose blocks are the circles of $\mathscr{C}$ which contain P, and whose incidence is that inherited from $\mathscr{I}$).

The *order* of $\mathscr{I}$ is the (common) order of the affine planes $\mathscr{I}_P$.

Inversive planes arise quite naturally in geometry in the following way. Let K be a skewfield and $\mathscr{O}$ an ovoid in $PG(3, K)$ (i.e., $\mathscr{O}$ is a set of points satisfying: (1) no three points of $\mathscr{O}$ are collinear, and (2) if $P \in \mathscr{O}$, then the union of all lines meeting $\mathscr{O}$ only in P is a plane). Then the following incidence structure, $\mathscr{I}(\mathscr{O})$, is an inversive plane:

Points of $\mathscr{I}(\mathscr{O})$ are the points of $\mathscr{O}$.

Circles of $\mathscr{I}(\mathscr{O})$ are those planes which meet $\mathscr{O}$ in more than one point.

Incidence is inclusion.

An inversive plane is said to be *egglike*[1] if it is isomorphic to some $\mathscr{I}(\mathscr{O})$. These play a role in the study of inversive planes similar to that of the Desarguesian planes in the theory of projective planes. In fact, the striking analogy between these two theories goes much further, and since it in large part motivates the work of this paper, I will dwell on it a little here.

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[1] This term is due to Dembowski and Hughes [4].

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The role of the Pappian planes is taken by the subclass of egglike planes for which K (as above) is commutative and $\mathcal{O}$ is a nonruled quadric. Van der Waerden and Smid [10] gave substance to this analogy by proving that the inversive planes belonging to this subclass are precisely those which satisfy the "theorem of Miquel" [3, p. 255]. This configurational proposition, then, corresponds to the theorem of Pappus.

In the same paper, the authors discuss a second axiom—the "Buschelsatz," or bundle theorem (see Section 2)—which holds in all egglike planes, and which is accordingly a candidate for the role of Desargues' theorem. The bundle theorem does not hold in all inversive planes (see [5], e.g.), but it does hold in all inversion geometries of dimension greater than two ([7, 9]). Mäurer's theorem [9], which implies that all such geometries are egglike[2], relies heavily on this fact. All of this is, of course, precisely analogous to the situation for projective planes and higher dimensional projective spaces.

The analogy described above is completed by the following theorem, whose proof is the object of this paper.

THEOREM. *All inversive planes which satisfy the bundle theorem are egglike.*

In an early attack on the problem, Hesselbach [6] was able to prove this for the special case of "topological" inversive planes. Recently (see [8], I was able to prove the theorem for finite inversive planes (actually for odd order planes, since Dembowski [1, 2] showed that *all* inversive planes of even order are egglike). Thus in the present paper the theorem is proved under the assumption that the inversive plane is infinite.[3]

The proof is accomplished by constructing a three-dimensional projective space in which the given inversive plane is suitably embedded. Section 2, most of which is copied from [8], shows how an inversive plane which satisfies the bundle theorem may be embedded in an incidence structure $\mathscr{S}$, which might be termed our first approximation to the desired projective space. In Section 3, we derive some of the easier properties of $\mathscr{S}$.

Although relatively simple, the two lemmas of Section 4, especially 4.3, exhibit what is perhaps the underlying idea of the proof. This involves exploiting connections between configurations in the "local" structure of lines and planes containing some given point, and corresponding configurations in the structure of $\mathscr{S}$ itself. In Sections 5 and 6, we obtain the lines (called "near-lines") of the space, the hard work being done in Section 5, which contains the most delicate part of the proof. Finally, in Section 7, we prove

[2] That is, the spheres of the inversion geometry are the intersections of an ovoid with flats in $PG(d, K)$.

[3] If the inversive plane is finite, the proof given here can probably be modified (and simplified considerably) to give an alternate (though not necessarily easier) proof of the result in [8].

that the system of points and near-lines satisfies "Pasch's axiom," and the theorem follows.

2. The Incidence Structure $\mathscr{S}$

An incidence structure (see [3, p. 1]) is usually defined as a triple $(\mathscr{P}, \mathscr{B}, I)$, with $\mathscr{P}$ a set of "points", $\mathscr{B}$ a set of "blocks", and $I \subseteq \mathscr{P} \times \mathscr{B}$ (or I a symmetric subset of $(\mathscr{P} \times \mathscr{B}) \cup (\mathscr{B} \times \mathscr{P})$). In what follows, we will frequently think of a block as a subset of the point set, and when $(P, b) \in I$, we will say "$P \in b$," "b contains P," etc. Incidence structures will be denoted simply $(\mathscr{P}, \mathscr{B})$.

As in the above paragraph, capital letters will always represent points; but notation for blocks will vary. Two sets (of points) are called *tangent* if they have intersection size one.

From now on $\mathscr{I} = (\mathscr{O}, \mathscr{C})$ is a (fixed) inversive plane of infinite order n.

Let $P \in \mathscr{O}$. Since tangency at P is the same as parallelism in $\mathscr{I}_P$, tangency at P is an equivalence relation on the circles through P. The equivalence classes are called *pencils* (at P). If $P \in x \in \mathscr{C}$, then $[P, x]$ denotes the pencil at P which contains x. If P, Q are distinct points of $\mathscr{O}$, the set of all circles containing both P and Q is called a *bundle* and denoted $[P, Q]$. A *fan* is a pencil or a bundle.

We say that $P \in \mathscr{O}$ *supports* pencils $[P, x]$ and bundles $[P, Q]$, and write $\boldsymbol{\beta}$ for the support of (i.e., the set of points supporting) the fan β. The circles of a fan β partition $\mathscr{O} \backslash \boldsymbol{\beta}$. Two fans are *compatible* if they have disjoint supports and contain a common circle. A circle x and fan β are *disjoint* if $x \cap \boldsymbol{\beta} = \varnothing$.

Following Dembowski [3, p. 255], we define a *4-chain* to be any quadruple x_1, x_2, x_3, x_4 of circles such that no three of the x_i contain a common point, but $x_i \cap x_{i+1} \neq \varnothing$, with subscripts taken mod 4. Define, for such a 4-chain, fans $\alpha_i = [x_i, x_{i+1}]$. We say that $\mathscr{I}$ satisfies the *bundle theorem* (BT) if the following condition holds:

> For any 4-chain x_1, x_2, x_3, x_4 with $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ as above, α_1 and α_3 are compatible iff α_2 and α_4 are compatible (Figure 1).

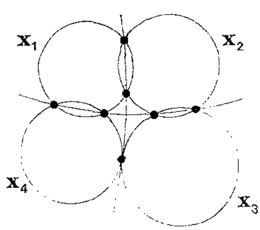

Figure 1

We define a *bundle complex* (BC) as a set Σ of fans satisfying:

Every $P \in \mathcal{O}$ supports exactly one fan of Σ (denoted $\Sigma(P)$). (2.1)

Any two distinct fans of Σ are compatible. (2.2)

A circle x is said to *belong* to a BC Σ if x is in some fan of Σ. From (2.2) we have:

If x belongs to Σ, then $x \in \Sigma(P)$ for every $P \in x$. (2.3)

From now on we assume that $\mathscr{I}$ satisfies BT. The next lemma and two corollaries are proved in [8].

LEMMA 2.4. *Let β_1 and β_2 be compatible fans. Then there is a unique BC containing β_1 and β_2.*

COROLLARY 2.5. *If x is a circle disjoint from a fan β, then there is a unique BC Σ such that $\beta \in \Sigma$ and x belongs to Σ.*

COROLLARY 2.6. *If X is a point, and β a fan, with $X \notin \boldsymbol{\beta}$, then there is a unique BC Σ such that $\beta \in \Sigma$ and $\Sigma(X)$ is a pencil.*

We define the incidence structure $\mathscr{S} = (\mathscr{P}, \mathscr{B})$ as follows:

$$\mathscr{P} = \mathcal{O} \cup \{\Sigma\colon \Sigma \text{ is a BC}\}.$$
$$\mathscr{B} = \{\tau_P\colon P \in \mathcal{O}\} \cup \{\langle x\rangle\colon x \in \mathscr{C}\}.$$

Incidence: τ_P is incident with P and with all BCs Σ for which $\Sigma(P)$ is a pencil. $\langle x\rangle$ is incident with all points of x and with all BC's Σ for which x belongs to Σ.

Members of $\mathscr{B}$ are called *planes* (rather than blocks). Planes τ_P are called *tangent planes*, and planes $\langle x\rangle$ are called *circle planes.*

3. PRELIMINARIES

DEFINITION. A *Δ-system* is a family of sets having pairwise the same intersection. This common intersection is called the *kernel* of the Δ-system.

As might be expected, our main task is to show that if $P \neq Q$ are in $\mathscr{P}$, then the set of planes containing P and Q is a Δ-system.

DEFINITIONS. If β is a fan, we define a *line*

$$l_\beta = \boldsymbol{\beta} \cup \{\Sigma\colon \Sigma \text{ is a BC}, \beta \in \Sigma\}.$$

l_β is called a *secant* if $|\boldsymbol{\beta}|(= |l_\beta \cap \mathcal{O}|) = 2$, and a *tangent* if $|\boldsymbol{\beta}| = 1$.

Remark 3.1. If $P \in \mathcal{O}$ and $P \neq Q \in \mathcal{P}$, then there is a unique line (denoted PQ) which contains P and Q. If $Q \in \mathcal{O}$, this line is $l_{[P,Q]}$; if Q is a BC Σ, the line is $l_{\Sigma(P)}$. On the other hand, if $P, Q \in \mathcal{P}\backslash\mathcal{O}$, $P \neq Q$, then by Lemma 2.4 there is at most one line containing P and Q.

PROPOSITION 3.2. (a) *If $l = l_\beta$ is any line and π any plane, then either $|l \cap \pi| = 1$, or $l \subseteq \pi$.*

(b) *The set of planes containing l is* (i) $\{\langle x\rangle : x \in \beta\}$ *if β is a bundle, and* (ii) $\{\langle x\rangle : x \in \beta\} \cup \{\tau_P\}$ *if β is a pencil with* $\boldsymbol{\beta} = \{P\}$.

Proof. First, if $x \in \beta$ then $l_\beta \subseteq \langle x\rangle$, and if β is a pencil $[P, y]$, then $\tau_P \supseteq l_\beta$. Both statements follow from the definitions.

If $\pi \cap \boldsymbol{\beta} = \varnothing$, then $|l \cap \pi| = 1$ by Corollary 2.5 (if π is a circle plane) or 2.6 (if π is a tangent plane). So $|l \cap \pi| \geqslant 1$ is always true.

If $P \in \pi \cap \boldsymbol{\beta}$ and π contains a second point, Q, of l, then either (i) $Q \in \boldsymbol{\beta} \Rightarrow \pi = \langle x\rangle$ with $x \in \beta \Rightarrow l \subseteq \pi$; or (ii) Q is a BC containing β. In this case: if π is a circle plane $\langle x\rangle$, we have $P \in x$ and x belongs to $\Sigma \Rightarrow x \in \Sigma(P) = \beta$ (by (2.3)) $\Rightarrow l \subseteq \pi$; and if $\pi = \tau_P$, then $\beta = \Sigma(P)$ is a pencil, so $l \subseteq \pi$.

We have thus shown that any plane which meets l twice is one of those listed in (b) (and that all of these contain l). ∎

PROPOSITION 3.3. *If P, Q, R are distinct points of $\mathcal{P}$, with $P \in \mathcal{O}$ and $PQ \neq PR$, then there is a unique plane* (*denoted* PQR) *which contains P, Q, and R.*

Proof. Let $PQ = l_\beta$ and $PR = l_\gamma$. Then $P \in \boldsymbol{\beta} \cap \boldsymbol{\gamma}$, and $\beta \neq \gamma$ (since $PQ \neq PR$). By 3.2(a), a plane contains P, Q, R iff it contains $l_\beta \cup l_\gamma$. If β and γ are not both pencils, then by Proposition 3.2(b), $PQR = \langle x\rangle$ where x is the unique circle of $\beta \cap \gamma$. If β and γ are both pencils, then $\beta \cap \gamma = \varnothing$, and $PQR = \tau_P$. ∎

COROLLARY 3.4. *If l is a line and $Q \in \mathcal{P}\backslash l$, then there is a unique plane* (*denoted Ql*) *which contains $\{Q\} \cup l$.* (*That is, the planes containing l partition $\mathcal{P}\backslash l$.*)

Proof. Let $P \in l \cap \mathcal{O}$ and $R \in l\backslash\{P\}$. Then by Proposition 3.2(a), a plane contains $\{Q\} \cup l$ iff it contains P, Q, and R. But there is exactly one such plane by Proposition 3.3. ∎

DEFINITION. The intersection of two distinct planes is called a 2-*line*.

PROPOSITION 3.5. *The lines are precisely the 2-lines which meet $\mathcal{O}$.*

Proof. Let $l = \pi_1 \cap \pi_2$ be a 2-line, $P \in l \cap \mathcal{O}$, and $Q \in l \backslash \{P\}$. Then $l = \pi_1 \cap \pi_2 \supseteq PQ$ by Proposition 3.2(a). If $R \in l \backslash PQ$, then $\{P, Q, R\} \subseteq \pi_1, \pi_2 \Rightarrow \pi_1 = PQR = \pi_2$ (by Proposition 3.3), a contradiction. So $l = PQ$.
On the other hand, any line is obviously contained in, hence equal to, some 2-line. ∎

PROPOSITION 3.6. *All 2-lines have the same cardinality.*

Proof. Let l be a 2-line $\pi_1 \cap \pi_2$. It is easy to see that there is a point P of $\mathcal{O}$ which is on exactly one of π_1, π_2 (say π_1). We show that there is a bijection between the points of l and the points of some line in the projective completion of $\mathcal{I}_P$. Since all $\mathcal{I}_Q$ have the same cardinality, this will be sufficient to prove the proposition.
By Proposition 3.2(a), the map

$$s\colon \pi_1 \cap \pi_2 \to \mathcal{L} := \{m\colon m \text{ is a line through } P \text{ in } \pi_1\},$$

defined by $s(Q) = PQ$, is a bijection. If $\pi_1 = \tau_P$, then $\mathcal{L} = \{l_\beta\colon \boldsymbol{\beta} = \{P\}\}$ is in 1–1 correspondence with the parallel classes of $\mathcal{I}_P$. If $\pi_1 = \langle x \rangle$, then $\mathcal{L} = \mathcal{L}' \cup \{l_{[P,x]}\}$, where $\mathcal{L}' := \{l_\beta\colon P \in \boldsymbol{\beta}, x \in \beta, \beta \text{ is a bundle}\}$ is in 1–1 correspondence with the points of the line $x \backslash \{P\}$ of $\mathcal{I}_P$. ∎

PROPOSITION 3.7. *Let π_1, π_2, π_3 be distinct planes. Then* $|(\pi_1 \cap \pi_2) \backslash \pi_3| = |(\pi_1 \cap \pi_3) \backslash \pi_2|$.

Proof. First suppose there exists $P \in \pi_1 \cap \mathcal{O}$ such that $P \notin \pi_2, \pi_3$. For $i = 2, 3$ the map $s_i\colon \pi_1 \cap \pi_i \to \{m\colon m \text{ is a line through } P \text{ in } \pi_1\}$, defined by $s_i(Q) = PQ$, is a bijection (again by Proposition 3.2(a)). In particular, $s_3^{-1}s_2$ induces a bijection of $(\pi_1 \cap \pi_2) \backslash \pi_3$ and $(\pi_1 \cap \pi_3) \backslash \pi_2$.
Now suppose there exists $P \in \pi_1 \cap \mathcal{O} \cap \pi_2$ (or π_3). Then $l = \pi_1 \cap \pi_2$ is a line by Proposition 3.5, so either $l \subseteq \pi_3$ or $|l \cap \pi_3| = 1$. If $l \subseteq \pi_3$, then $\pi_1 \cap \pi_2 = l$ (by 3.5), and $|(\pi_1 \cap \pi_2) \backslash \pi_3| = |(\pi_1 \cap \pi_3) \backslash \pi_2| = 0$. If $|l \cap \pi_3| = 1$, i.e., $|\pi_1 \cap \pi_2 \cap \pi_3| = 1$, then the result follows from Proposition 3.6. ∎

COROLLARY 3.8. *If π_1, π_2, π_3 are distinct planes with $\pi_1 \cap \pi_2 \subseteq \pi_3$, then $\{\pi_1, \pi_2, \pi_3\}$ is a Δ-system.*

COROLLARY 3.9. *If l and m are respectively a line and a 2-line contained in a plane π, then $l \cap m \neq \varnothing$.*

Proof. Let $m = \pi_2 \cap \pi_3 \subseteq \pi$. Then $\varnothing \neq l \cap \pi_2 = l \cap \pi \cap \pi_2 = l \cap \pi_2 \cap \pi_3$ (by 3.8) $= l \cap m$. ∎

PROPOSITION 3.10. *Let π_1, π_2, π_3 be distinct planes, at least one of which is a circle plane. If $|(\pi_1 \cap \pi_2) \backslash \pi_3| < \infty$, then $\{\pi_1, \pi_2, \pi_3\}$ is a Δ-system.*

Proof. Since $|(\pi_i \cap \pi_j)\backslash\pi_k|$ is constant for $\{i, j, k\} = \{1, 2, 3\}$, we may assume π_1 is a circle plane. By Proposition 3.8, it is enough to show $\pi_1 \cap \pi_3 \subseteq \pi_2$. Let $l_0 = \pi_1 \cap \pi_2 \cap \pi_3$, $l_2 = (\pi_1 \cap \pi_2)\backslash\pi_3$, $l_3 = (\pi_1 \cap \pi_3)\backslash\pi_2$, and $l = l_0 \cup l_2 = \pi_1 \cap \pi_2$.

Suppose there exists $A \in l_3$. If $p \in \pi_1 \cap \mathcal{O}$, $P \notin \pi_3$, then by Corollary 3.9, $PA \cap l \neq \varnothing$. But $PA \cap l_0 = \varnothing$, since otherwise π_3 contains two points of PA, and so contains P by Proposition 3.2(a). Thus $PA \cap l_2 \neq \varnothing$. But there are infinitely many distinct lines PA of this type, and they must meet l_2 in infinitely many distinct points (since, by Remark 3.1, any two meet only in A). This is contrary to hypothesis, so we must have $l_3 = \varnothing$, which is what we wanted to show. ∎

DEFINITION. A *near-line* is a 2-line (m) which satisfies the following two equivalent conditions:

(N1) Any plane containing at least two points of m contains m.

(N2) For all $P \in \mathcal{O}\backslash m$, there is a (necessarily unique) plane Pm which contains $\{P\} \cup m$.

(That (N1) $\Rightarrow$ (N2) is immediate from 3.3. Conversely, suppose m satisfies (N2), and let π be a plane containing points Q, R of m. Let $P \in \pi \cap \mathcal{O}$. If $P \in m$, then m is a line by Proposition 3.5, and satisfies (N1) by Proposition 3.2(a). If $P \notin m$, then both π and the plane of (N2) contain P, Q, R, and must therefore be equal. (See Proposition 3.3).)

Remark 3.11. Every line is a near-line (by Propositions 3.5 and 3.2(a)).

Remark 3.12. If l is a near-line, then the set of planes meeting l at least twice (i.e., containing l) is a Δ-system with kernel l. (For let $l = \pi_1 \cap \pi_2$, and suppose π_3 and π_4 are distinct planes containing l. Then if we assume $\pi_3 \neq \pi_1$, we may apply Corollary 3.8 (twice) to show $l = \pi_1 \cap \pi_2 = \pi_1 \cap \pi_3 = \pi_3 \cap \pi_4$.)

In particular, there is at most one near-line containing a given pair of points X, Y. If this near-line exists we denote it XY.

PROPOSITION 3.13. *If m is a near-line and π a plane, then either $|m \cap \pi| = 1$ or $m \subseteq \pi$.*

Proof. We just have to show $m \cap \pi \neq \varnothing$. Suppose $P \in \pi \cap \mathcal{O}$ (and $P \notin m$). Then by Corollary 3.9, $\varnothing \neq (Pm \cap \pi) \cap m \subseteq \pi \cap m$. ∎

COROLLARY 3.14. *If m and t are respectively a near-line and a 2-line contained in a plane π, then $m \cap t \neq \varnothing$.*

Proof. Same as proof of Corollary 3.9.

4. Two Lemmas

Definition. In a projective plane π let (C, U, V, W, Z, Z_1) be a 6-tuple of distinct points, and k, l lines satisfying

$$\begin{aligned} C, Z, Z_1 &\in k, \\ U, V, W &\in l, \\ Z, Z_1 &\notin l, \\ V, W &\notin k. \end{aligned} \tag{4.1}$$

We say that π is *Desarguesian with respect to* (C, U, V, W, Z, Z_1) if for each $Y \in WZ\backslash\{W, Z\}$, $Y \notin CU$, we have $Y_1 \in WZ_1$, where Y_1 is defined by:

$$\begin{aligned} X &:= UY \cap VZ, \\ X_1 &:= CX \cap VZ_1, \\ Y_1 &:= UX_1 \cap CY. \end{aligned} \tag{4.2}$$

(Note that X, X_1 and Y_1 represent points, not sets. This particular notational abuse will be common from now on.)

Definition. For $P \in \mathcal{O}$, we denote by $\mathcal{S}/P$ the incidence structure whose points and lines are respectively the lines and planes of $\mathcal{S}$ which contain P, incidence being inclusion. $\mathcal{S}/P$ is easily seen to be a projective plane. Elements of $\mathcal{S}/P$ are denoted by upper bars, using capital letters for points and lower case letters for lines. (We allow some ambiguity: $\bar{X}$ is used both for a point of $\mathcal{S}/P$ and for a line through P (in $\mathcal{S}$), and similarly for $\bar{l}$. The intended meaning should usually be clear from the context.)

Let (P, C, U, V, W, Z, Z_1) be a 7-tuple of distinct points of $\mathcal{P}$ satisfying

$$\begin{array}{ll} \text{(i)} & P, C, U, V \in \mathcal{O}; \\ \text{(ii)} & C, Z, Z_1 \text{ lie on a line } k; \\ \text{(iii)} & U, V, W \text{ lie on a line } l; \\ \text{(iv)} & k \cap l = \phi; \\ \text{(v)} & P \text{ does not lie on any of the planes } UWZ_1, CWZ_1, UWZ, VZZ_1; \\ \text{(vi)} & \{CWZ, UWZ, PWZ\} \text{ is a } \Delta\text{-system.} \end{array} \tag{4.3}$$

Define points $\bar{C} = PC$, $\bar{U} = PU, \ldots, \bar{Z}_1 = PZ_1$, and lines $\bar{k} = Pk$, $\bar{l} = Pl$ of $\mathcal{S}/P$. It is straightforward to check that $\bar{C}$, $\bar{U}$, $\bar{V}$, $\bar{W}$, $\bar{Z}$, $\bar{Z}_1$ are distinct, and that the 6-tuple $(\bar{C}, \bar{U}, \bar{V}, \bar{W}, \bar{Z}, \bar{Z}_1)$ together with $\bar{k}$ and $\bar{l}$ satisfies conditions (4.1). ($\bar{C} \neq \bar{U}$ because a line cannot contain three points of $\mathcal{O}$. The first two conditions of (4.1) come from (ii) and (iii). The other conditions and the remaining inequalities between the points are consequences of (v).)

LEMMA 4.4 (with notation as above). *$\mathcal{S}/P$ is Desarguesian with respect to $(\bar{C}, \bar{U}, \bar{V}, \bar{W}, \bar{Z}, \bar{Z}_1)$ iff $\{CWZ_1, UWZ_1, PWZ_1\}$ is a Δ-system.*

(Note: Both of the notations CWZ and CWZ_1 will be used in what follows, although they represent the same plane. Similar abuses will, unfortunately, occur throughout the paper.)

Proof. Let $m = CWZ \cap UWZ$ ($\subseteq \overline{WZ}$ by (vi)). If to each $Y \in m$ we correspond the point $\bar{Y} := PY$ of $\mathcal{S}/P$, then by Corollary 3.9 we have a bijection of m and (the line) $\overline{WZ}$. Denote by Y_0 the point of m corresponding to the intersection point, $\bar{Y}_0$, of $\overline{WZ}$ and $\overline{CU}$.

If for each $Y \in m$ we let Y_1 be the unique point of $CY \cap UWZ_1$, then $Y \leftrightarrow Y_1$ is, according to Corollary 3.9, a bijection of m and $CWZ_1 \cap UWZ_1$ (noting that $CWZ_1 = Cm$). Note that $W \leftrightarrow W$ and $Z \leftrightarrow Z_1$.

We can obtain the restriction of this bijection to $m \backslash \{W, Z\}$ in another way: For $Y \in m\backslash\{W, Z\}$, let X be the intersection point of the lines UY and VZ (which meet since they both lie in the plane UWZ). Let X_1 be the intersection of the lines CX and VZ_1 (which lie in the plane VZZ_1). Finally, the lines UX_1 and CY lie in the plane CUY, so they meet in a point which, since it lies on $UX_1 \subseteq UWZ_1$, can only be Y_1.

Now starting with $(\bar{C}, \bar{U}, \bar{V}, \bar{W}, \bar{Z}, \bar{Z}_1)$ and $\bar{Y} := PY$, $Y \neq Y_0$, define the points $\bar{X}$, $\bar{X}_1$ and $\bar{Y}_1$ of $\mathcal{S}/P$ according to (4.2). Then we have:

$$\begin{aligned} X &= UY \cap VZ \subseteq \overline{UY} \cap \overline{VZ} := \bar{X}; \\ X_1 &= CX \cap VZ_1 \subseteq \overline{CX} \cap \overline{VZ}_1 := \bar{X}_1; \\ Y_1 &= UX_1 \cap CY \subseteq \overline{UX}_1 \cap \overline{CY} := \bar{Y}_1. \end{aligned}$$

In particular, $Y_1 \in \bar{Y}_1$, so that $\bar{Y}_1 \in \overline{WZ}_1$ is equivalent to $Y_1 \in PWZ_1$. This is enough to prove the lemma, since:

$$\begin{aligned} &\mathcal{S}/P \text{ is Desarguesian with respect to } (\bar{C}, \bar{U}, \bar{V}, \bar{W}, \bar{Z}, \bar{Z}_1) \\ &\quad \Leftrightarrow \forall\ \bar{Y} \in \overline{WZ}\backslash\{\bar{W}, \bar{Z}, \bar{Y}_0\},\ \bar{Y}_1 \in \overline{WZ}_1 \\ &\quad \Leftrightarrow \forall\ Y \in m\backslash\{W, Z, Y_0\},\ Y_1 \in PWZ_1 \\ &\quad \Leftrightarrow PWZ_1 \supseteq (CWZ_1 \cap UWZ_1)\backslash\{(Y_0)_1\} \\ &\quad \Leftrightarrow \{CWZ_1, UWZ_1, PWZ_1\} \text{ is a } \Delta\text{-system (by Proposition 3.10).} \quad \blacksquare \end{aligned}$$

LEMMA 4.5. *Let (P, C, W, X, S, T, Y) be a 7-tuple of distinct points and h a secant (to $\mathcal{O}$) satisfying:*

(a) $P, C \in \mathcal{O}$;

(b) *Each of the triples* $\{P, X, T\}, \{P, Y, S\}, \{C, X, S\}, \{C, T, Y\}$ *is collinear*;

(c) $C \notin PX$ (i.e., $\{P, C, X, S, T, Y\}$ *contains no collinear triples other than those in* (b));

(d) $W \notin PCX$;

(e) $W \in h$, *h meets none of the lines* PX, PY, CX, CY;

(f) $\{CWX, PWX, Xh\}$ *is a* Δ*-system*;

(g) $\{CWT, PWT, Th\}$ *is a* Δ*-system*;

then $\{CWS, PWS, Sh\}$ *is a* Δ*-system iff* $\{CWY, PWY, Yh\}$ *is a* Δ*-system.*

Proof. If $W \in \mathcal{O}$, then the result follows from Remark 3.12. (In this case both triples are Δ-systems.) Suppose, then, that $W \notin h \cap \mathcal{O} = \{U, V\}$. Since $|h \cap PCX| = 1$, we may assume $V \notin PCX$. Then it is straightforward to verify that each of the 7-tuples (P, C, U, V, W, X, S) and (P, C, U, V, W, T, Y) satisfies the hypotheses (4.3). We also observe that if we define $\bar{X} = PX$, $\bar{S} = PS$, $\bar{T} = PT$ and $\bar{Y} = PY$ in $\mathcal{S}/P$, then $\bar{X} = \bar{T}$ and $\bar{S} = \bar{Y}$. Thus a double application of Lemma 4.4 yields:

$$\begin{aligned} &\{CWS,\ UWS = Sh,\ PWS\} \text{ is a } \Delta\text{-system} \\ &\quad \Leftrightarrow \mathcal{S}/P \text{ is Desarguesian with respect to } (\bar{C}, \bar{U}, \bar{V}, \bar{W}, \bar{X}, \bar{S}) \\ &\qquad = (\bar{C}, \bar{U}, \bar{V}, \bar{W}, \bar{T}, \bar{Y}) \\ &\quad \Leftrightarrow \{CWY,\ UWY = Yh,\ PWY\} \text{ is a } \Delta\text{-system.} \end{aligned}$$ ∎

5. Some Δ-Systems

Let $\pi = \langle x \rangle$ be a circle plane, and suppose that X and Y are points of π, each of which lies on only finitely many secants in π. Then there are infinitely many (so at least two) points $Z \in x$ for which $XZ = \pi \cap \tau_Z = YZ$. Thus $X = Y$.

Definition. If π is a circle plane, and if there is a point of π lying on only finitely many secants in π, then this point is denoted $B(\pi)$.

By the preceding remarks, we have

$$\text{if } B(\pi) \text{ exists, then it is unique (for given } \pi). \tag{5.1}$$

Suppose W is a point of $\mathcal{P}\backslash\mathcal{O}$ which lies on the three noncoplanar secants a, b, c, where $a \cap \mathcal{O} = \{E, F\}$, $b \cap \mathcal{O} = \{G, H\}$, and $P \in c \cap \mathcal{O}$. Let $A = EH \cap GF$ (the intersection being nonempty since both lines lie in the plane

WEG), and $l = PA$. All of these elements we now fix for the duration of Section 5.

Let Z be any point of $l\backslash\{A\}$, and d any secant on Z such that d meets neither of GA, EA. (5.2)

LEMMA 5.3 (with Z and d as above). (a) $\{EZW, GZW, Wd\}$ *is a* Δ-*system.* (b) *If* $Z \neq B(PAW)$, *then* $\{EZW, GZW, PAW\}$ *is a* Δ-*system.*

Proof. If d meets GW or EW, then (a) is trivial. Otherwise, the 7-tuple of distinct points (G, E, Z, A, H, F, W) and the secant d are easily (tediously) seen to satisfy conditions (a)–(e) of Lemma 4.5. In fact, (f) and (g) also hold: $ZA(=PA)$ and ZF being lines (since $P, F \in \mathcal{O}$), it follows from Remark 3.12 that $\{EZA, GZA, Ad\}$ and $\{EZF, GZF, Fd\}$ are Δ-systems. Moreover, $\{EZH, GZH, Hd\}$ is a Δ-system (ZH is a line), so that Lemma 4.5 implies (a).

Now $PAW \neq ZGA, ZEA$, so that there are at most two secants through Z in PAW which violate (5.2). Thus if $Z \neq B(PAW)$ there is a secant through Z in PAW satisfying (5.2), and (b) follows from (a). ∎

DEFINITION. For any plane π containing PW, define

$$\mathscr{H}(\pi) = \{h\colon h \text{ is a secant through } W;\ h \nsubseteq \pi, PAW;\ h \cap \mathcal{O} \nsubseteq PAE \cup PAG\}.$$

While reading the rest of Section 5, the reader may find it helpful to refer to Fig. 2.

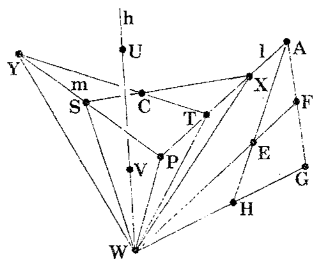

FIGURE 2

Let π be any plane containing PW except PAW. Let m be a line of π other than PW which contains P and satisfies:

$$E, G \notin \pi_0 := lm, \tag{5.4}$$

$$\pi_0 \text{ is a circle plane.} \tag{5.5}$$

Such a line clearly exists; in fact, there are at most three choices of m which fail to satisfy (5.4) or (5.5), since (i) $E \in \pi_0 \Leftrightarrow \pi_0 = El \Leftrightarrow m = \pi \cap El$ (these two planes being distinct since $W \in El$ would imply $P \in WEG$); (ii) $G \in \pi_0 \Leftrightarrow m = \pi \cap Gl$ (as (i)); and (iii) π_0 is not a circle plane $\Rightarrow m = \tau_P \cap \pi$ ($\pi \neq \tau_P$ since $W \notin \tau_P$).

LEMMA 5.6. *Let π, m be as above, and $Y \in m$. Assume $Y \neq B(\pi_0)$. Then for all $h, h' \in \mathscr{H}(\pi)$, $Yh \cap \pi = Yh' \cap \pi$.*

Proof. If W, Y are collinear, this follows from Remark 3.12, so in particular we may assume $Y \neq P$.

The lemma will follow if we can show

if $h \in \mathscr{H}(\pi)$, then for all but finitely many $C \in \pi_0 \cap \mathscr{O}$ either

(i) $\{\pi, CWY, Yh\}$ is a Δ-system, or $\qquad (*)$

(ii) CY is a tangent.

For there are infinitely many $C \in \pi_0 \cap \mathscr{O}$ such that CY is a secant (since $Y \neq B(\pi_0)$). So if $h, h' \in \mathscr{H}(\pi)$, there must exist $C \in (\pi_0 \cap \mathscr{O}) \backslash m$ for which $\{\pi, CWY, Yh\}$ and $\{\pi, CWY, Yh'\}$ are Δ-systems, and for such a C we have $Yh \cap \pi = \pi \cap CWY = Yh' \cap \pi$.

Fix $h \in \mathscr{H}(\pi)$ and let $h \cap \mathscr{O} = \{U, V\}$. Define

$$\mathscr{F} = \{P, A\} \cup (\tau_U \cap l) \cup (\tau_V \cap l) \cup \{B(PAW)\}$$

if $B(PAW)$ exists and is on l, and

$$\mathscr{F} = \{P, A\} \cup (\tau_U \cap l) \cup (\tau_V \cap l)$$

otherwise. Then $\mathscr{F}$ is a finite subset of l, since $P \in l$ implies $|l \cap \tau_U| = |l \cap \tau_V| = 1$.

Let C be any point of $\pi_0 \cap \mathscr{O}$ which satisfies

$$CY \text{ is not a tangent,} \tag{5.7}$$

$$C \notin l, m, Yh, \tag{5.8}$$

$$T := CY \cap l \notin \mathscr{F}. \tag{5.9}$$

Note there are at most $5 + 2|\mathscr{F}| < \infty$ choices of $C \in \pi_0 \cap \mathscr{O}$ which are excluded by (5.8) and (5.9). Thus $(*)$ will be proved if we can show

for all C satisfying (5.7)–(5.9), $\{\pi, CWY, Yh\}$ is a Δ-system. $\qquad (**)$

Choose $S \in m\backslash\{P, Y\}$ such that

$$W, S \text{ are collinear,} \tag{5.10}$$

$$CS \text{ is a secant,} \tag{5.11}$$

$$X := CS \cap l \notin \mathscr{F}, \tag{5.12}$$

$$S \notin Ch. \tag{5.13}$$

Note that there are infinitely many $S \in m\backslash\{P, Y\}$ satisfying (5.10), since $W \notin \tau_P$ implies π is a circle plane. Since at most $1 + |\mathscr{F}| + 1 < \infty$ of these are eliminated by (5.11)–(5.13), we can choose S as desired.

We now verify that (P, C, W, X, S, T, Y) and h satisfy the hypotheses of Lemma 4.5. That the points of the 7-tuple are distinct is fairly obvious, as are conditions 4.5(a)–(c). If $W \in PCX$ $(= \pi_0)$, then $m = \pi_0 \cap \pi = PW$, contrary to the choice of m, so we have (d). Condition (e) follows from (5.8) and (5.13).

In view of (5.7), (5.9), (5.11), and (5.12), we can obtain (f) and (g) by repeated applications of Corollary 3.8, once we have shown

if $Z \in l\backslash\mathscr{F}$ and CZ is a secant, then $\{EWZ, GWZ, \pi_1\}$ is a Δ-system if π_1 is any one of PWZ, CWZ, Zh. (***)

When $\pi_1 = PWZ$ $(= PAW)$, this follows from Lemma 5.3(b).

When $\pi_1 = CWZ$: If (the secant) CZ meets GA in a point J, then A, $J \in \pi_0 \Rightarrow G \in \pi_0$, contrary to (5.4). Thus, $CZ \cap GA = \varnothing$, and similarly $CZ \cap EA = \varnothing$. (***) (with $\pi_1 = CWZ$) now follows from Lemma 5.3(a).

When $\pi_1 = Zh$: Since $Z \notin \mathscr{F}$, UZ and VZ are both secants. If UZ (VZ) meets EA, then $U(V) \in PAE$, and similarly if UZ (VZ) meets GA, then $U(V) \in PAG$. But at least one of U, V (say U) is on neither of PAE, PAG (by the definition of $\mathscr{H}(\pi)$), so we have $UX \cap GA = UX \cap EA = \varnothing$. Again Lemma 5.3(a) implies (***).

This proves (***).

We now have all hypotheses of Lemma 4.5. Moreover, that $\{CWS, PWS\ Sh\}$ is a Δ-system follows from Remark 3.12, since W and S are collinear (see (5.10)). Thus according to Lemma 4.5, $\{CWY, PWY = \pi, Yh\}$ is a Δ-system. This is the statement of (**), and completes the proof of the lemma. ∎

We now extend Lemma 5.6 to all points of $\pi\backslash\{W\}$.

LEMMA 5.14. *Let π be any plane containing PW except PAW, and let $Y \in \pi\backslash\{W\}$. Then for all $h, h' \in \mathscr{H}(\pi)$, $Yh \cap \pi = Yh' \cap \pi$.*

Proof. We first observe that

for any $h \in \mathscr{H}(\pi)$, the 2-lines $Yh \cap \pi$ with $Y \in m$ partition the points of $\pi \backslash \{W\}$. (5.15)

For any $Z \in \pi \backslash \{W\}$ is on a unique plane Zh (by Corollary 3.4) and the 2-line $Zh \cap \pi$ meets m by Corollary 3.9.

Suppose $Y \in m$, $Y \neq B(\pi_0)$, and let $Y' \in \pi \backslash \{W\}$ with $Y' \in Yh \cap \pi$ for some $h \in \mathscr{H}(\pi)$. Then $Y' \in Yh' \cap \pi$ for every $h' \in \mathscr{H}(\pi)$, by Lemma 5.6, and for all $h, h' \in \mathscr{H}(\pi)$ we have

$$Y'h \cap \pi = Yh \cap \pi = Yh' \cap \pi = Y'h' \cap \pi. \tag{5.16}$$

Now suppose $Y_0 := B(\pi_0)$ (exists and) lies on m. Let $h, h' \in \mathscr{H}(\pi)$, and suppose there exists $Y' \in (Y_0 h' \cap \pi) \backslash (Y_0 h \cap \pi)$. Then by (5.15) there is some $Y \in m \backslash \{Y_0\}$ with $Y' \in Yh \cap \pi$. But then, using (the last equality of) (5.16), we have $Y_0 h' \cap \pi = Y'h' \cap \pi = Yh' \cap \pi$, contradicting $Y \in m \backslash \{Y_0\}$. Thus $Y_0 h' \cap \pi = Y_0 h \cap \pi$ for all $h, h' \in \mathscr{H}(\pi)$, and the argument leading to (5.16) may now be repeated with Y_0 replacing Y to complete the proof of the lemma. ∎

Now let $Y \in PAW \backslash PW$, and $h, h' \in \mathscr{H}(PAW)$ (so $h, h' \not\subseteq PAW$, but they may lie in any of the other planes through PW). We assert that if $Yh \neq Yh'$, then $Yh \cap Yh' \subseteq PAW$ (which by Corollary 3.8 implies $\{Yh, Yh', PAW\}$ is a Δ-system). Suppose this is not the case, that is there exists $Z \in (Yh \cap Yh') \backslash PAW$. Then $h, h' \in \mathscr{H}(PWZ)$. (For $h \notin \mathscr{H}(PWZ)$ can only occur if $h \subseteq PWZ$ —see definition of $\mathscr{H}(\pi)$—which would imply $Yh = Zh = Ph$. But then $h \subseteq PWY$, contradicting $h \in \mathscr{H}(PWY = PAW)$.) It follows form Lemma 5.14 that $\{Zh, Zh', PWZ\}$ is a Δ-system. But this is impossible, since $Y \in (Zh \cap Zh') \backslash PWZ$. (Note that $Zh = Yh \neq Yh' = Zh'$.) This proves our assertion. We have shown

LEMMA 5.17. *Let π be any plane containing PW, and let $Y \in \pi \backslash \{W\}$. Then for all $h, h' \in \mathscr{H}(\pi)$, $Yh \cap \pi = Yh' \cap \pi$.*

COROLLARY 5.18. *If $Y \in \mathscr{P} \backslash PW$, then the (distinct) planes of $\{PWY\} \cup \{Yh\colon h \in \mathscr{H}(PWY)\}$ form a Δ-system.*

Now if $Y \in \mathscr{P} \backslash PW$, and if $\pi \neq PWY$ is a plane containing W and Y, then at most finitely many of the secants through W in π are not members of $\mathscr{H}(PWY)$. To see this, observe that at most one such secant lies in PWY and at most one in PAW (neither of these being equal to π), and that at most two such secants (h) have $h \cap \mathscr{O} \subseteq PAE \cup PAG$ (since these planes contain at most two points apice of $\pi \cap \mathscr{O}$). In particular, if π is a circle plane with $W \neq B(\pi)$, then π will contain some secant $h \in \mathscr{H}(PWY)$. We thus have

COROLLARY 5.19. *If $Y \neq W$, then*

$$\{\pi\colon \pi \text{ is a circle plane on } W, Y;\ W \neq B(\pi)\}$$

is a Δ-system.

Remarks. 1. For $Y \in PW$, 5.19 is a consequence of Remark 3.12.

2. While Corollary 5.19 is somewhat weaker than Corollary 5.18, it will be sufficient for our purposes from now on, and will allow us to dispense with the cumbersome structure built up in this section.

6. NEAR-LINES

DEFINITIONS. 1. For distinct $W, Y \in \mathscr{P}$, let

$$\mathscr{K}_{WY} = \{\pi\colon \pi \text{ is a circle plane on } W, Y;\ W \neq B(\pi)\},$$
$$\mathscr{C}_{WY} = \{\pi\colon \pi \text{ is a circle plane on } W, Y\}\ (= \mathscr{C}_{YW}).$$

2. Let

$$\mathscr{O}\!\ell = \{X \in \mathscr{P}\colon \text{there is a finite collection of planes on } X \text{ whose union contains all secants through } X\}.$$

For each $X \in \mathscr{O}\!\ell$ we choose (perhaps arbitrarily) a finite set of planes, $\mathscr{D}(X)$, which is minimal with respect to the property that the union of its planes contains all secants through X.

Remark 6.1. If $X \in \mathscr{O}\!\ell$ and π is a circle plane on X, then either $X = B(\pi)$ or $\pi \in \mathscr{D}(X)$.

We state an even weaker version of Corollary 5.19.

COROLLARY 6.2. *If W, Y are distinct points with $W \notin \mathscr{O}\!\ell$, then $\mathscr{K}_{WY}$ is a Δ-system.*

Proof. If $W \in \mathscr{O}$, this is a consequence of Remark 3.12. Otherwise, we observe that Section 5, and in particular Corollary 5.19, assumes only that ($W \notin \mathscr{O}$ and) W lies on three noncoplanar secants. This will surely be the case whenever $W \notin \mathscr{O}\!\ell$. ∎

COROLLARY 6.3. *If W, Y are distinct points of $\mathscr{P}\backslash\mathscr{O}\!\ell$ with $|\mathscr{K}_{WY} \cap \mathscr{K}_{YW}| \geq 2$, then $\mathscr{C}_{WY}$ is a Δ-system.*

Proof. $\mathscr{C}_{WY} = \mathscr{K}_{WY} \cup \mathscr{K}_{YW}$ (by (5.1)), and $\mathscr{K}_{WY}$, $\mathscr{K}_{YW}$ are Δ-systems. ∎

Our goal is to show that any two points of $\mathscr{P}$ lie on a near-line. Before doing this we need some concrete information about $\mathscr{O}\!\ell$.

LEMMA 6.4. (a) *Any plane which contains at least two points of $\mathfrak{A}$ contains $\mathfrak{A}$.*

(b) *If* $|\mathfrak{A}| \geqslant 2$ *then at most finitely many of the planes in* (a) *are circle planes.*

Proof. Let π be a circle plane and $X, Y \in \mathfrak{A} \cap \pi$, $X \neq Y$. Since $X = B(\pi) = Y$ is impossible (see (5.1)), we must have $\pi \in \mathscr{D}(X) \cup \mathscr{D}(Y)$ by 6.1. (Thus Lemma 6.4(b) will follow once we have Lemma 6.4(a).) The number of distinct planes of the form Yh with h a secant, $X \in h$, $Y \notin h$, is therefore finite. If, in particular, π is a circle plane on X and $X \neq B(\pi)$, then $\mathfrak{A} \subseteq \pi$. But if a circle plane π contains the distinct points X, Y of $\mathfrak{A}$, then at least one of $X \neq B(\pi)$, $Y \neq B(\pi)$ holds, and the preceding comment implies $\mathfrak{A} \subseteq \pi$.

Finally, suppose $P \in \mathcal{O}$ and $X, Y \in \mathfrak{A} \cap \tau_P$. If for some $Z \in \mathfrak{A}$, PZ is a secant, then $X \notin PZ$ implies PZX is a circle plane, and therefore contains $\mathfrak{A}$. But then $\{P, X, Y\} \subseteq PZX \cap \tau_P$, so that XY is a line lying on τ_P. Now suppose Q is another point of $\mathcal{O}$ with $X, Y \in \tau_Q$. (Such a point clearly exists, since we already know that only finitely many circle planes contain X and Y.) But then $P \in XY \subseteq \tau_Q$ is a contradiction. We conclude $\mathfrak{A} \subseteq \tau_P$, which proves (a). ∎

LEMMA 6.5. *Let* $W \neq Y, \{W, Y\} \nsubseteq \mathfrak{A}$. *Then* (a) *$W$, Y lie on a near-line WY.* (b) $|WY \cap \mathfrak{A}| \leqslant 1$.

Proof. Since each secant containing W or Y lies in a plane of $\mathscr{C}_{WY}$, and since $\{W, Y\} \nsubseteq \mathfrak{A}$, $\mathscr{C}_{WY}$ must be infinite. But $\mathscr{C}_{WY} = \mathscr{K}_{WY} \cup \mathscr{K}_{YW}$ (by (5.1)), so without loss of generality $\mathscr{K}_{WY}$ is infinite. Then $W \notin \mathfrak{A}$ (by Remark 6.1), so $\mathscr{K}_{WY}$ is a Δ-system (by Corollary 6.2). Let $l =$ kernel $\mathscr{K}_{WY}$. Note that $\mathscr{K}_{WX} = \mathscr{K}_{WY}$ for each $X \in l \backslash \{W\}$. Since $\mathscr{K}_{WY}$ is infinite, Lemma 6.4 gives us

$$|l \cap \mathfrak{A}| \leqslant 1. \tag{6.6}$$

Thus (b) will be true once we have proved (a).

First, suppose there exists $X \in l \backslash \mathfrak{A} \backslash \{W\}$ such that $\mathscr{C}_{WX}$ is not a Δ-system. Then according to Corollary 6.3, we must have $X = B(\pi)$ for all but at most one $\pi \in \mathscr{K}_{WX}$. Call this plane π_0 if it exists. Then for all $Z \in l \backslash \{W, X\}$ we must have $(\mathscr{K}_{WY} \backslash \{\pi_0\} =) \mathscr{K}_{WX} \backslash \{\pi_0\} \subseteq \mathscr{K}_{WZ} \cap \mathscr{K}_{ZW}$. This implies $Z \notin \mathfrak{A}$ (by Remark 6.1), and consequently (by Corollary 6.3) that

> there is a finite subset $\mathscr{E}$ of l, with $W \in \mathscr{E}$, such that for all $Z \in l \backslash \mathscr{E}$, $\mathscr{C}_{WZ}$ is a Δ-system (with kernel l). (6.7)

(In the present instance $\mathscr{E} = \{W, X\}$.)

On the other hand, if $\mathscr{C}_{WX}$ is a $\varDelta$-system for every $X \in l\backslash\mathfrak{A}\backslash\{W\}$, then, according to (6.6), we again have (6.7) (with $\mathscr{E} = \{W\} \cup (\mathfrak{A} \cap l)$).

We may thus assume that (6.7) holds. Let $P \in \mathscr{O}\backslash l$. If $l\backslash\mathscr{E} \subseteq \tau_P$, then $l \subseteq \tau_P$ by Proposition 3.10. If, on the other hand, there is some $V \in l\backslash\mathscr{E}$ for which PV is secant, then $PWV \in \mathscr{C}_{WV}$, which implies $l \subseteq PWV$ (by (6.7)). Thus l satisfies (N2) and is a near-line. ∎

Finally we assert that if $|\mathfrak{A}| \geqslant 2$, then $\mathfrak{A}$ itself is a near-line. By Lemma 6.4(a) this will follow if $\mathfrak{A}$ is a 2-line. But if π, π' are distinct planes containing $\mathfrak{A}$, then by Lemma 6.5(b), $\pi \cap \pi' \subseteq \mathfrak{A}$, i.e., $\pi \cap \pi' = \mathfrak{A}$. We have shown

THEOREM 6.8. *Any two points of $\mathscr{S}$ lie on a near-line.*

7. PLANES

(We may now write XY for the near-line containing X and Y.)

LEMMA 7.1. *If in $\mathscr{S}|P$ ($P \in \mathscr{O}$) the 6-tuple of distinct points ($\bar{C}$, $\bar{U}$, $\bar{V}$, $\bar{W}$, $\bar{Z}$, $\bar{Z}_1$) and the distinct lines $\bar{k}$, $\bar{l}$ satisfy* (4.1), *and if $\bar{C}$, $\bar{U}$, $\bar{V} \notin \bar{l}_\infty$ ($:= \tau_P$), then $\mathscr{S}|P$ is Desarguesian with respect to ($\bar{C}$, $\bar{U}$, $\bar{V}$, $\bar{W}$, $\bar{Z}$, $\bar{Z}_1$).*

Proof. By Theorem 6.8 and Remark 3.12 it is enough to find C, U, V, W, Z, $Z_1 \in \mathscr{P}$ such that $\bar{C} = PC,\ldots$, $\bar{Z}_1 = PZ_1$, and (P, C, U, V, W, Z, Z_1) satisfies conditions (4.3i–v) (i.e., condition (vi) and the fact that $\{CWZ_1, UWZ_1, PWZ_1\}$ is a $\varDelta$-system are guaranteed by Theorem 6.8 and Remark 3.12). To do this, choose $C = \bar{C} \cap (\mathscr{O}\backslash\{P\})$, $U = \bar{U} \cap (\mathscr{O}\backslash\{P\})$ and $V = \bar{V} \cap (\mathscr{O}\backslash\{P\})$. (All of these points exist since $\bar{C}$, $\bar{U}$, $\bar{V} \notin \bar{l}_\infty$). Let $W = UV \cap \bar{W}$. Finally, let k be any line through C in $\bar{k}$ other than PC and $\bar{k} \cap CUV$, and let $Z = \bar{Z} \cap k$, $Z_1 = \bar{Z}_1 \cap k$. Then conditions (4.3i–iv) are obvious, and (v) follows from the last two conditions of (4.1). ∎

COROLLARY 7.2. *$\mathscr{S}|P$ is Desarguesian for all $P \in \mathscr{O}$.*

Proof. It is a simple consequence of Lemma 7.1 that $\mathscr{S}|P$ is ($\bar{C}$, $\bar{l}$)-Desarguesian (see [3, p. 122]) whenever $\bar{C} \notin \bar{l}_\infty$ and $\bar{l} \neq \bar{l}_\infty$ (since the latter condition guarantees that at least two of the points on the axis in any given configuration will not lie on $\bar{l}_\infty$). For $\bar{C} \notin \bar{l}_\infty$, let $\bar{B} \in \bar{l}_\infty$ and let $\bar{l}$, $\bar{m}$ be two other lines through $\bar{B}$. Then $\mathscr{S}|P$ is ($\bar{C}$, $\bar{l}$)- and ($\bar{C}$, $\bar{m}$)-Desarguesian, hence ($\bar{C}$, $\bar{l}_\infty$)-Desarguesian (see [3, p. 123]). So $\mathscr{S}|P$ is ($\bar{C}$, $\bar{l}$)-Desarguesian whenever $\bar{C} \notin \bar{l}_\infty$. A similar argument now shows that $\mathscr{S}|P$ is ($\bar{C}$, $\bar{l}$)-Desarguesian in all cases. For given $\bar{C}$, $\bar{l}$ with $\bar{C} \in \bar{l}_\infty$, let $\bar{m} \neq \bar{l}$, $\bar{l}_\infty$ be a line on $\bar{C}$ and $\bar{A}$, $\bar{B}$ two additional points of $\bar{m}$. Then $\mathscr{S}|P$ is ($\bar{A}$, $\bar{l}$)- and ($\bar{B}$, $\bar{l}$)-Desarguesian, hence ($\bar{C}$, $\bar{l}$)-Desarguesian (again, see [3, p. 123]). ∎

It is well known (see [3, p. 27–28]) that an incidence structure $\mathscr{R} = (\mathscr{X}, \mathscr{L})$ is isomorphic to the point–line incidence structure of $PG(d, K)$ for some d and some skewfield K, provided it satisfies the following three axioms:

$$\text{Any two distinct } X, Y \in \mathscr{X} \text{ are on exactly one } l \in \mathscr{L}. \tag{7.3}$$

$$|\, l \,| \geqslant 3 \text{ for each } l \in \mathscr{L}. \tag{7.4}$$

If a, b, l, m are distinct elements of $\mathscr{L}$, no three containing a common point, if a meets b, and if each of a, b meets each of l, m, then l meets m. (7.5)

LEMMA 7.6. *The point, near-line structure of $\mathscr{S}$ satisfies* (7.3)–(7.5).

Proof. Axiom (7.3) is the same as Theorem 6.8, while (7.4) follows from Proposition 3.6. We must show (7.5).

Let a and b be near-lines meeting in U. Let l and m be near-lines with $a \cap l = V$, $a \cap m = W$, $b \cap l = X_1$, $b \cap m = Y_1$, and $|\{U, V, W, X_1, Y_1\}| = 5$. We must show that l and m intersect. We may assume

the only coplanar triples from $\{U, V, W, X_1, Y_1\}$ are $\{U, V, W\}$ and $\{U, X_1, Y_1\}$, (7.7)

since otherwise the whole configuration is easily seen to lie in a plane and the result follows from Corollary 3.14.

Let $Z \in \mathscr{O}$. Since $|\tau_Z \cap \{V, W, X_1, Y_1\}| \leqslant 2$, we may assume (w.l.o.g.) that ZW is a secant, meeting $\mathscr{O}$ in a second point Y. From (7.7) we have $W \notin \mathscr{O}$ (so $W \neq Y, Z$), and $ZW \cap b = ZW \cap l = \varnothing$. The lines UY and VZ intersect (in X say), since they lie in the plane YUW. By Corollary 3.14, the near-lines YY_1 and XX_1 intersect (say in C), since they lie in the plane YUY_1.

Let $ZC = k$. We find that $k \cap l \neq \varnothing$ since both near-lines are in ZVX_1, and $k \cap m \neq \varnothing$ since both near-lines are in YWY_1. Let $k \cap l = Z_1$ and $k \cap m = Z_2$.

Choose $P \in \mathscr{O}$ such that P is not on any of the planes YUW, YUY_1, YWY_1, ZVX_1. Note that (7.7) implies Pa, Pb, Pl and Pm are distinct planes. Now if in $\mathscr{S}/P$ we put $\bar{U} = PU$, $\bar{V} = PV$, etc., then, in light of Corollary 7.2, the lines $\bar{Z}\bar{C}$, $\bar{V}\bar{X}_1$, and $\bar{W}\bar{Y}_1$ must be concurrent. But this gives

$$PZ_1 = \bar{Z}\bar{C} \cap \bar{V}\bar{X}_1 = \bar{Z}\bar{C} \cap \bar{W}\bar{Y}_1 = PZ_2.$$

If $Z_1 \neq Z_2$, then $P \in Z_1Z_2 = ZC \subseteq YWY_1$, contrary to assumption. We conclude that $Z_1 = Z_2$ is the required intersection point of l and m. ∎

We have now shown that the structure of points and near-lines of $\mathscr{S}$ is (isomorphic to) the point-line structure of some $PG(d, K)$. If π is one of our original planes, then π contains any near-line which it meets in at least two points (definition of near-line), and any two near-lines in π intersect (by Corollary 3.14), so that π is also a plane of the projective geometry $\mathscr{S}$. But then the fact that π meets every near-line (see Proposition 3.13) implies that the dimension, d, of $\mathscr{S}$ is 3.

The near-lines which meet $\mathcal{O}$ are, by Proposition 3.5, precisely the lines l_β, so they meet $\mathcal{O}$ in at most two points apiece. By Proposition 3.2(b), a line containing $P \in \mathcal{O}$ is tangent to $\mathcal{O}$ if and only if it lies in τ_P. Thus $\mathcal{O}$ is an ovoid. Finally, the planes (of the projective geometry) which meet $\mathcal{O}$ are precisely the planes of $\mathscr{B}$ (the ones we started with). Thus the plane sections of $\mathcal{O}$ of size greater than one are precisely the circles of $\mathscr{I}$, and the proof of the theorem is complete.

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